Quantisation of derived Poisson structures

QUANTISATION OF DERIVED POISSON STRUCTURES

J.P.PRIDHAM

Abstract. We prove that every 0-shifted Poisson structure on a derived Deligne–Mumford n-stack admits a curved A_∞ quantisation whenever the stack has perfect cotangent complex; in particular, this applies to LCI schemes. Where the Kontsevich–Tamarkin approach to quantisation hinges on invariance of the Hochschild complex under affine transformations, we instead exploit the observation that it carries an involution, and that such involutive deformations of the complex of polyvectors are essentially unique.

Introduction

For smooth algebraic varieties in characteristic 0, Kontsevich showed in [Kon2] that all Poisson structures admit DQ algebroid quantisations. Via local choices of connections, the question reduced to constructing quantisations of affine space. These could then be handled as in [Tam, Kon1, VdB]: formality of the E_2 operad associates to the Hochschild complex a deformation of the P_2-algebra of multiderivations, and invariance under affine transformations ensures that it is the unique deformation.

We now consider generalisations of this question to singular varieties and to derived stacks, considering quantisations of 0-shifted Poisson structures in the sense of [Pri4, CPT$^+$]. For positively shifted structures, the analogous question is a formality, following from the equivalence $E_{n+1} \simeq P_{n+1}$ of operads. Quantisations for non-degenerate 0-shifted Poisson structures were established in [Pri2], and we now consider degenerate quantisations, addressing the remaining unsolved case of [Toë, Conjecture 5.3].

The construction of non-degenerate quantisations in [Pri2, Pri3] only relied on the fact that the Hochschild complex is an involutive deformation of the complex of multiderivations. Our strategy in this paper is closer to [Tam, Kon1] in that we establish an equivalence between the two complexes. As in [Pri2, Pri3], the key observation is still that the Hochschild complex of a CDGA carries an involution corresponding to the endofunctor on deformations sending an algebra to its opposite. For a suitable choice of formality quasi-isomorphism for the E_2 operad, which can be deduced from the action of the Grothendieck–Teichmüller group (§2.2), the Hochschild complex becomes an involutive deformation of the P_2-algebra of multiderivations.

We show (Theorem 1.17) that such deformations are essentially unique whenever the complexes of polyvectors and of multiderivations are quasi-isomorphic. This is satisfied when the CDGA has perfect cotangent complex, so gives an equivalence between polyvectors and the Hochschild complex (Theorem 2.10). For derived Deligne–Mumford stacks with perfect cotangent complex, this yields quantisations of 0-shifted Poisson structures (Corollary 2.12), which take the form of curved A_∞ deformations of the étale structure sheaf.

Notation and terminology. We write CDGAs (commutative differential graded algebras) and DGAAs (differential graded associative algebras) as chain complexes (homological grading), and denote the differential on a chain complex by δ. Our conventions for shifts of chain and cochain complexes are that $(V_{[n]})_i := V_{n+i}$ and $(V^{[n]})^i := V^{n+i}$.

Given a DGAA A, and A-modules M, N in chain complexes, we write $\underline{\mathrm{Hom}}_A(M, N)$ for the chain complex given by

$$\underline{\mathrm{Hom}}_A(M, N)_i = \mathrm{Hom}_{A_\#}(M_\#, N_{\#[i]}),$$

with differential $\delta f = \delta_N \circ f \pm f \circ \delta_M$, where $V_\#$ denotes the graded vector space underlying a chain complex V.

Contents

1. Involutively filtered deformations of Poisson algebras

In this section, we characterise a certain class of filtered deformations of Poisson algebras equipped with involutions. We initially work in a very general setting, in order to obtain results we can immediately apply to quantisation problems for diagrams, and with a view to possible future generalisations. Of the results in the first two subsections, readers might find it easier initially just to concentrate on Definition 1.6, Lemma 1.10 and Remark 1.11, which suffice for applications to derived affine schemes.

We will assume that all filtrations are increasing and exhaustive, unless stated otherwise. Given a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-vector space V, we write $\mathcal{W}_i V$ for the summand of weight i, and define a weight filtration W by setting $W_n V := \bigoplus_{i \leqslant n} \mathcal{W}_i V$. For $\mathbb{G}_m$-equivariant complexes U, V, we write $\mathcal{W}_i \underline{\mathrm{Hom}}(U, V)$ for the complex $\prod_j \underline{\mathrm{Hom}}(\mathcal{W}_j U, \mathcal{W}_{i+j} V)$ of homomorphisms of weight i, with similar conventions for complexes of derivations etc.

1.1. Involutively filtered deformations of $\mathcal{P}$-algebras.

Definition 1.1. We say that a vector space V is involutively filtered if it is equipped with a filtration W and an involution e which preserves W and acts on $\mathrm{gr}^W_i V$ as multiplication by $(-1)^i$.

Observe that if V is involutively filtered, then the involution gives an eigenspace decomposition $V = V^{e=1} \oplus V^{e=-1}$, and because $\mathrm{gr}^W_{i-1} V^{e=(-1)^i} = 0$, we have $W_{2j+1} V^{e=1} = W_{2j} V^{e=1}$ and $W_{2j} V^{e=-1} = W_{2j-1} V^{e=-1}$.

Definition 1.2. Define a $\mathbb{G}_m$-equivariant dg Hopf algebra $\mathbb{Q}[\eth]$ over $\mathbb{Q}$ by taking $\eth$ of homological degree -1 and of weight -2 for the $\mathbb{G}_m$ action, commutativity forcing $\eth^2 = 0$. The comultiplication $\mathbb{Q}[\eth] \to Q[\eth] \otimes_{\mathbb{Q}} \mathbb{Q}[\eth]$ is defined by $\eth \mapsto \eth \otimes 1 + 1 \otimes \eth$.

This is slightly different from the dg Hopf algebra $\mathbb{Q}[\textit{đ}]$ of [Pri3, §1.1.1], in which $\textit{đ}$ has weight -1; the difference corresponds to the data of an involution, and the results below hold without involutions if we replace $\eth$ with $\textit{đ}$. Also beware that [Pri3] considered decreasing filtrations, so its weights have the opposite signs to ours.

Definition 1.3. For a complete involutively filtered chain complex (V, W_iV, e) over $\mathbb{Q}$, we define a $\mathbb{G}_m$-equivariant $\mathbb{Q}[\eth]$-module $\mathfrak{gr}^{W,e}V$ to be given in weight i by

$$\begin{aligned} \mathfrak{gr}_i^{W,e}V &:= \{v \in \mathrm{cone}(W_{i-1}V \to W_iV) \ : \ e(v) = (-1)^i v\}, \\ &= \{v \in \mathrm{cone}(W_{i-2}V \to W_iV) \ : \ e(v) = (-1)^i v\} \end{aligned}$$

with $\eth \colon \mathfrak{gr}_i^W V \to \mathfrak{gr}_{i-2}^W V_{[-1]}$ given by the identity on $W_{i-2}V_{[-1]}$ (and necessarily 0 elsewhere).

This gives an equivalence of ∞-categories from the category of complete filtered $\mathbb{Q}$-chain complexes localised at filtered quasi-isomorphisms to the category of $\mathbb{G}_m$-equivariant $\mathbb{Q}[\eth]$-modules in chain complexes localised at quasi-isomorphisms. The homotopy inverse functor is given by

$$\bigoplus_i \mathcal{W}_i E \mapsto (\bigoplus_{i>0} \mathcal{W}_i E \oplus \prod_{i \leqslant 0} \mathcal{W}_i E, \delta \pm \eth),$$

equipped with the complete exhaustive filtration

$$W_i := (\prod_{j \leqslant i} \mathcal{W}_j E, \delta \pm \eth)$$

and involution $e(\sum_i v_i) = \sum_i (-1)^i v_i$ for $v_i \in \mathcal{W}_i E$.

Defining a symmetric monoidal structure on $\mathbb{Q}[\eth]$-modules by $\otimes_{\mathbb{Q}}$, with $\eth$ acting on $M \otimes_{\mathbb{Q}} N$ via the comultiplication on $\mathbb{Q}[\eth]$, the functor $\mathfrak{gr}$ and its homotopy inverse above are both lax monoidal.

Definition 1.4. Given a $\mathbb{G}_m$-equivariant dg operad $\mathcal{P}$ over $\mathbb{Q}$ and a dg Hopf algebra B over $\mathbb{Q}$, define the operad $\mathcal{P} \circ B$ by $(\mathcal{P} \circ B)(n) := \mathcal{P}(n) \otimes B^{\otimes n}$, with operad structure defined by the distributive law $B \otimes \mathcal{P}(n) \to \mathcal{P}(n) \otimes B^{\otimes n}$ given by the comultiplication on B.

If we define a filtration W on a $\mathbb{G}_m$-equivariant dg operad $\mathcal{P}$ by setting $W_i\mathcal{P}$ to be spanned by terms of weight $\leqslant i$ for the $\mathbb{G}_m$-action, and an involution e given by the action of $-1 \in \mathbb{G}_m$, then the functor $\mathfrak{gr}^{W,e}$ above gives an equivalence from the ∞-category of $(\mathcal{P}, W, e)$-algebras A in complete filtered $\mathbb{Q}$-chain complexes to the ∞-category of $\mathcal{P} \circ \mathbb{Q}[\eth]$-algebras in $\mathbb{G}_m$-equivariant $\mathbb{Q}$-chain complexes. If we forget the $\eth$-action, note that $\mathfrak{gr}^{W,e}A$ is quasi-isomorphic to the $\mathbb{G}_m$-equivariant $\mathcal{P}$-algebra $\bigoplus_i \mathrm{gr}_i^W A$.

We thus make the following definition:

Definition 1.5. Given a $\mathbb{G}_m$-equivariant coloured dg operad (i.e. a dg multicategory) $\mathcal{A}$ over $\mathbb{Q}$ and a $\mathbb{G}_m$-equivariant operad $\mathcal{P}$ over $\mathbb{Q}$, define the space of complete involutively filtered derived $(\mathcal{P}, W, e)$-algebras in $\mathcal{A}$ to be the space

$$\mathrm{map}(\mathcal{P} \circ \mathbb{Q}[\eth], \mathcal{A})$$

of maps in the ∞-category of $\mathbb{G}_m$-equivariant coloured dg operads over $\mathbb{Q}$.

Here, the ∞-category of dg operads is defined by localising at morphisms $f\colon \mathcal{A} \to \mathcal{B}$ which induce quasi-isomorphisms $\mathcal{A}(x_1, \ldots, x_n; y) \to \mathcal{A}(fx_1, \ldots, fx_n; fy)$ and equivalences on the underlying homotopy categories (objects are colours, with morphisms $\mathrm{H}_0\mathcal{A}(x, y)$).

In practice, we will only ever apply Definition 1.5 to multicategories underlying symmetric monoidal dg categories $\mathcal{C}$, in which case a morphism $\mathcal{P} \circ \mathbb{Q}[\eth] \to \mathcal{A}$ consists of an object $C \in \mathcal{C}$ together with a morphism

$$\{\mathcal{P}(n) \otimes \mathbb{Q}[\eth]^{\otimes n}\}_n \to \{\underline{\mathrm{Hom}}_{\mathcal{C}}(C^{\otimes n}, C)\}_n$$

of dg operads.

1.2. **Almost commutative Poisson algebras.** We now consider non-unital P_k-algebras (i.e. $(k-1)$-shifted Poisson algebras); these are non-unital CDGAs equipped with a Lie bracket of chain degree $k-1$ acting as a biderivation. They are governed by an operad P_k which can be written as $\mathrm{Com} \circ (s^{1-k}\mathrm{Lie})$ via a distributive law (cf. [LV, §8.6]), for the operads $\mathrm{Com}, \mathrm{Lie}$ governing non-unital commutative algebras and Lie algebras, where $(s\mathcal{P})(n) := \mathcal{P}(n)_{[n-1]}$.

Definition 1.6. Define the $\mathbb{G}_m$-equivariant dg operad P_k^{ac} to be the dg operad $\mathrm{Com} \circ s^{1-k}\hbar\mathrm{Lie}$, where $(\hbar\mathcal{P})(i) := \hbar^{i-1}\mathcal{P}(i)$ for any operad $\mathcal{P}$ and $\hbar$ has degree 0 and weight 1 for the $\mathbb{G}_m$-action.

Define an almost commutative involutive P_k-algebra over a CDGA R to be an $(R \otimes P_k, W, e)$-algebra A in involutively complete filtered R-chain complexes, where $W_i P_k$ is spanned by terms of weight $\leqslant i$ for the $\mathbb{G}_m$-action on P_k^{ac}.

Thus a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra is a P_k-algebra equipped with a $\mathbb{G}_m$-action for which multiplication has weight 0 and the Lie bracket has weight -1. An almost commutative involutive P_k-algebra is a P_k-algebra equipped with a complete filtration W satisfying $W_i \cdot W_j \subset W_{i+j}$ and $[W_i, W_j] \subset W_{i+j-1}$, together with an involution $*$ preserving the filtration, satisfying $(a \cdot b)^* = a^* \cdot b^*$ and $[a, b]^* = -[a^*, b^*]$, and acting as $(-1)^i$ on gr_i^W. Such algebras automatically give rise to complete filtered derived (P_k^{ac}, W, e)-algebras in the sense of Definition 1.5, which we thus refer to as almost commutative involutive derived P_k-algebras.

Definition 1.7. Given a differential graded Lie algebra (DGLA) L with homological grading, define the the Maurer–Cartan set by

$$\mathrm{MC}(L) := \{\omega \in L_{-1} \ \mid \delta\omega + \frac{1}{2}[\omega, \omega] = 0 \in \bigoplus_n L_{-2}\}.$$

Following [Hin2], define the Maurer–Cartan space $\underline{\mathrm{MC}}(L)$ (a simplicial set) of a nilpotent DGLA L by

$$\underline{\mathrm{MC}}(L)_n := \mathrm{MC}(L \otimes_{\mathbb{Q}} \Omega^\bullet(\Delta^n)),$$

where

$$\Omega^\bullet(\Delta^n) = \mathbb{Q}[t_0, t_1, \ldots, t_n, \delta t_0, \delta t_1, \ldots, \delta t_n]/(\sum t_i - 1, \sum \delta t_i)$$

is the commutative dg algebra of de Rham polynomial forms on the n-simplex, with the t_i of degree 0.

Given an inverse system $L = \{L_\alpha\}_\alpha$ of nilpotent DGLAs, define

$$\mathrm{MC}(L) := \varprojlim_\alpha \mathrm{MC}(L_\alpha) \quad \underline{\mathrm{MC}}(L) := \varprojlim_\alpha \underline{\mathrm{MC}}(L_\alpha).$$

Note that $\mathrm{MC}(L) = \mathrm{MC}(\varprojlim_\alpha L_\alpha)$, but $\underline{\mathrm{MC}}(L) \neq \underline{\mathrm{MC}}(\varprojlim_\alpha L_\alpha)$.

By [Cav, Theorem 4.22] (following [Hin1, BM]), there is a model structure on $\mathbb{G}_m$-equivariant dg operads over $\mathbb{Q}$ in which all objects are fibrant.

Lemma 1.8. *For the operadic cobar construction* Ω *of* [LV, §6.5.5]*, cofibrant replacements of* P_k^{ac} *and* $P_k^{ac} \circ \mathbb{Q}[\eth]$ *are given by*

$$\Omega(s^k\hbar^{-1}P_k^{ac})^\vee \quad \text{and} \quad \Omega(\mathbb{Q}[\hbar^2] \circ s^k\hbar^{-1}P_k^{ac})^\vee.$$

Proof. The shifted analogue of [LV, Theorem 7.4.6] shows that $\Omega\mathcal{P}^{¡}$ is a cofibrant replacement for a Koszul dg operad $\mathcal{P}$, where as in [LV, 7.2.3], the Koszul dual co-operad $\mathcal{P}^{¡}$ is given by $\mathcal{P}^{¡} = (s(\mathcal{P}^!))^\vee$, for the Koszul dual operad $\mathcal{P}^!$. We have $\mathrm{Lie}^! = \mathrm{Com}$, $\mathrm{Com}^! = \mathrm{Lie}$ and $\mathbb{Q}[\eth]^! = \mathbb{Q}[\hbar^2]$, and the proof of [LV, 8.6.11] then adapts to this shifted setting (cf. [Fre, Appendix C]) to give $(P_k^{ac})^!$ as

$$\begin{aligned}
(P_k^{ac})^! &= (\mathrm{Com} \circ s^{1-k}\hbar\mathrm{Lie})^! \\
&= (s^{1-k}\hbar\mathrm{Lie})^! \circ \mathrm{Com}^! \\
&\cong (s^{k-1}\hbar^{-1}\mathrm{Com}) \circ \mathrm{Lie} \\
&= s^{k-1}\hbar^{-1}(\mathrm{Com} \circ (s^{1-k}\hbar\mathrm{Lie})) \\
&= s^{k-1}\hbar^{-1}P_k^{ac},
\end{aligned}$$

with

$$(P_k^{ac} \circ \mathbb{Q}[\eth])^! = \mathbb{Q}[\eth]^! \circ (P_k^{ac})^! = \mathbb{Q}[\hbar^2] \circ s^{k-1}\hbar^{-1}P_k^{ac}.$$

$\square$

Definition 1.9. Given a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra in a dg operad $\mathcal{A}$ (i.e. a $\mathbb{G}_m$-equivariant morphism $f\colon P_k^{ac} \to \mathcal{A}$), define the $\mathbb{G}_m$-equivariant DGLA $\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A) = \bigoplus_i \mathcal{W}_i\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A)$ to be the convolution complex

$$\begin{aligned}
&(\prod_n (s^k\hbar^{-1}P_k^{ac})(n) \otimes^{S_n} \mathcal{A}(n), \delta + [f\circ\alpha, -]) \\
= &(\prod_n (\hbar^{1-k}(P_k^{ac}(n))_{[k(n-1)]} \otimes^{S_n} \mathcal{A}(n), \delta + [f\circ\alpha, -])
\end{aligned}$$

where the bracket is defined via the convolution product of [LV, 6.4.4] (which is of weight 0 for the $\mathbb{G}_m$-action), the map $\alpha\colon (s^k\hbar^{-1}P_k^{ac})^\vee \to P_k^{ac}$ is defined by Koszul duality for P_k, and products are taken in the category of $\mathbb{G}_m$-equivariant complexes.

When $\mathcal{A}$ comes from an object A in a symmetric monoidal pre-triangulated dg category containing representations of $\mathbb{G}_m$, then the datum f above defines a P_k^{ac}-algebra structure on A, and the bar construction of [LV, 11.2] gives a $(s^k\hbar^{-1}P_k^{ac})$-coalgebra $\mathrm{B}_\alpha A$, with $\Omega_\alpha\mathrm{B}_\alpha A$ a quasi-free resolution of A. The DGLA $\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A)$ above then corresponds to the $\mathbb{G}_m$-equivariant complex of $(s^k\hbar^{-1}P_k^{ac})$-coalgebra coderivations on $\mathrm{B}_\alpha A$.

Lemma 1.10. *Given a* $\mathbb{G}_m$*-equivariant* P_k^{ac}*-algebra* A *in a* $\mathbb{G}_m$*-equivariant dg operad* $\mathcal{A}$ *(i.e. a* $\mathbb{G}_m$*-equivariant morphism* $P_k^{ac} \to \mathcal{A}$*), the space*

$$\mathrm{map}(P_k^{ac} \circ \mathbb{Q}[\eth], \mathcal{A}) \times^h_{\mathrm{map}(P_k^{ac}, \mathcal{A})} \{A\}$$

of almost commutative involutive derived P_k-algebras A' in $\mathcal{A}$ with $\mathrm{gr}^W A' \simeq A$ is given by the Maurer–Cartan space

$$\varprojlim_r \underline{\mathrm{MC}}(\prod_{j=1}^r \mathcal{W}_{-2j}\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A)).$$

Proof. By Lemma 1.8, we may realise the mapping spaces as spaces of morphisms from the cofibrant replacements $\Omega(s^k\hbar^{-1}P_k^{ac})^\vee$ and $\Omega(\mathbb{Q}[\hbar^2] \circ s^k\hbar^{-1}P_k^{ac})^\vee$ of P_k^{ac} and $P_k^{ac} \circ \mathbb{Q}[\eth]$ to the fibrant simplicial resolution $\Omega^\bullet(\Delta^*) \circ \mathcal{A}$ of $\mathcal{A}$. As in [LV, §6.5], the set of morphisms from the cobar construction is given by the Maurer–Cartan set of the convolution DGLA, realising the desired mapping space as

$$(\varprojlim_r \underline{\mathrm{MC}}(\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A)[\hbar^2]/\hbar^{2r})^{\mathbb{G}_m}) \times^h_{\underline{\mathrm{MC}}(\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A))^{\mathbb{G}_m}} \{0\},$$

which simplifies to the expression above, since $\hbar$ has weight 1. □

Remark 1.11. If we were looking at all almost commutative P_k-algebras instead of just the involutive ones, then we would replace $\eth$ with $\bar{d}$ and $\hbar^2$ with $\hbar$ in the reasoning above, giving the space $\varprojlim_r \underline{\mathrm{MC}}(\prod_{j=1}^r \mathcal{W}_{-j}\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A))$. This is unsurprising, since modifying the differential δ_A by a derivation in this space will still preserve the filtration, and not affect the associated graded algebra. The restriction to terms of even weight in Lemma 1.10 ensures that the modified differential $\delta \pm \eth$ on the filtered complex $\bigoplus_{i>0} \mathcal{W}_i A \oplus \prod_{i\leqslant 0} \mathcal{W}_i A$, commutes with the involution $-1 \in \mathbb{G}_m$.

Corollary 1.12. *Given a $\mathbb{G}_m$-equivariant P_k^{ac}-algebra A in a $\mathbb{G}_m$-equivariant dg operad $\mathcal{A}$ for which*

$$\mathrm{H}_i \mathcal{W}_{-2j}\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A) = 0$$

for all $j \geqslant 1$ and $i \geqslant -2$, the space of almost commutative involutive derived P_k-algebras A' in $\mathcal{A}$ with $\mathrm{gr}^W A' \simeq A$ is contractible.

Proof. By Lemma 1.10, it suffices to show that each map

$$\underline{\mathrm{MC}}(\prod_{j=1}^r \mathcal{W}_{-2j}\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A)) \to \underline{\mathrm{MC}}(\prod_{j=1}^{r-1} \mathcal{W}_{-2j}\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A))$$

is a weak equivalence, but as in [Pri4, Proposition 1.29], this map can be expressed as the homotopy fibre of a fibration over $\underline{\mathrm{MC}}(\mathcal{W}_{-2r}\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A))_{[-1]})$, whose ith homotopy group is $\mathrm{H}_{i-2}\mathcal{W}_{-2r}\mathbf{R}\mathcal{D}er_{P_k^{ac}}(A) = 0$ by hypothesis, making it contractible. □

Remark 1.13. If we take an involutively filtered dg operad $\mathcal{P}$ with an involutive equivalence $\mathrm{gr}^W \mathcal{P} \simeq P_k^{ac}$, then the conditions of Corollary 1.12 also ensure that the space of almost commutative involutive derived $\mathcal{P}$-algebras A' in $\mathcal{A}$ with $\mathrm{gr}^W A' \simeq A$ is contractible. This is because, although the controlling DGLA is defined in terms of $\mathbf{R}\mathcal{D}er_{\mathcal{P}}$ rather than $\mathbf{R}\mathcal{D}er_{P_k^{ac}}$, the associated graded pieces are the same.

1.3. **Uniqueness of deformations.** We now fix a CDGA R over $\mathbb{Q}$.

Writing $\mathrm{CoS}^p_B(M) := \mathrm{CoSymm}^p_B(M) = (M^{\otimes_B p})^{\Sigma_p}$, and expanding out Definition 1.9 in terms of the cotangent complex gives:

Lemma 1.14. *Given an I-diagram B of $\mathbb{G}_m$-equivariant P_k^{ac}-algebras in R-chain complexes, there are canonical $\mathbb{G}_m$-equivariant quasi-isomorphisms*

$$\mathcal{W}_i\mathbf{R}\mathcal{D}er_{P_k^{ac},R,I}(B) \simeq (\prod_{p\geqslant 1} \mathbf{R}\mathcal{W}_{i+1-p}\underline{\mathrm{Hom}}_B(\mathbf{L}\mathrm{CoS}^p_B((\mathbf{L}\Omega^1_{B/R})_{[-k]}), B), \delta + [\varpi, -])_{[-k]},$$

giving quasi-isomorphisms of $\mathbb{G}_m$-equivariant DGLAs on taking $\bigoplus_i$, where ϖ denotes the bivector on B corresponding to the Poisson bracket, and the right-hand complex is given the Schouten–Nijenhuis bracket.

Proposition 1.15. *If B is a non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebra over a CDGA R for which the map $(\mathcal{W}_1\mathbf{L}\Omega^1_{B/\mathcal{W}_0 B})\otimes_{\mathcal{W}_0 B} B \to \mathbf{L}\Omega^1_{B/\mathcal{W}_0 B}$ is a quasi-isomorphism, then $\mathcal{W}_i\mathbf{R}\mathcal{D}er_{P_k^{ac},R}(B)\simeq 0$ for $i\leqslant -2$.*

Proof. Without loss of generality, we may assume that B is cofibrant. We then have an exact triangle

$$\Omega^1_{\mathcal{W}_0 B}\otimes_{\mathcal{W}_0 B} B \to \Omega^1_B \to \Omega^1_{B/\mathcal{W}_0 B} \to \Omega^1_{\mathcal{W}_0 B}\otimes_{\mathcal{W}_0 B} B_{[-1]},$$

which by hypothesis simplifies to

$$\Omega^1_{\mathcal{W}_0 B}\otimes_{\mathcal{W}_0 B} B \to \Omega^1_B \to (\mathcal{W}_1\Omega^1_{B/\mathcal{W}_0 B})\otimes_{\mathcal{W}_0 B} B \to \Omega^1_{\mathcal{W}_0 B}\otimes_{\mathcal{W}_0 B} B_{[-1]}.$$

Since Ω^1_B is quasi-isomorphic to a B-module freely generated by terms of weights $0,1$, and since the weights of B are non-negative, we thus have that $\mathbf{R}\mathcal{W}_i\underline{\mathrm{Hom}}_B(\mathbf{L}\mathrm{CoS}^p_B((\mathbf{L}\Omega^1_{B/R})_{[-k]}),B)$ is acyclic for $i<-p$. The statement now follows from the description of Lemma 1.14. □

Definition 1.16. We say that a morphism $B\to C$ of $\mathbb{G}_m$-equivariant CDGAs over R is homotopy formally étale if it induces a quasi-isomorphism

$$\mathbf{L}\Omega^1_{B/R}\otimes^{\mathbf{L}}_B C \to \mathbf{L}\Omega^1_{C/R}$$

on cotangent complexes.

Theorem 1.17. *If B is a non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebra over a CDGA R, for which the map $(\mathcal{W}_1\mathbf{L}\Omega^1_{B/\mathcal{W}_0 B})\otimes_{\mathcal{W}_0 B} B \to \mathbf{L}\Omega^1_{B/\mathcal{W}_0 B}$ is a quasi-isomorphism, then the space of almost commutative involutive derived P_k-algebras (B',W) over R with fixed involutive equivalence $\mathrm{gr}^W B'\simeq B$ is contractible. We thus have an essentially unique equivalence $B'\simeq B$ of involutive filtered P_k-algebras for each such (B',W), the involution acting on $\mathcal{W}_i$ as $(-1)^i$.*

Moreover, for any homotopy formally étale morphism $f\colon B\to C$ of non-negatively weighted $\mathbb{G}_m$-equivariant P_k^{ac}-algebras over R, the space of morphisms $g\colon B\to C$ of almost commutative involutive P_k-algebras over R with $\mathrm{gr}^W g\simeq f$ is also contractible.

Proof. Proposition 1.15 implies that $\mathcal{W}_{-2j}\mathbf{R}\mathcal{D}er_{P_k^{ac},R}(B)$ is acyclic for all $j\geqslant 1$, so Corollary 1.12 shows that the space of almost commutative involutive deformations of B is contractible.

If we now write D for the [1]-diagram $(B\to C)$, then the argument of [Pri4, Lemma 2.3] shows that the restriction map

$$\mathbf{R}\mathcal{D}er_{P_k^{ac},R,[1]}(D)\to \mathbf{R}\mathcal{D}er_{P_k^{ac},R,[1]}(B)$$

is a $\mathbb{G}_m$-equivariant quasi-isomorphism, so the same argument shows that the space of involutive almost commutative deformations of the diagram is also contractible. □

Remark 1.18. If we were to consider non-involutive deformations instead, then the analogue of Theorem 1.17 would not hold. The non-involutive analogue of Corollary 1.12 involves homology of $\mathbf{R}\mathcal{D}er_{P_k^{ac},R}(B)$ in all negative weights, and the weight -1 term is given under the hypotheses of Theorem 1.17 by

$$(\prod_{p\geqslant 1}\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{W}_0 B}(\mathbf{L}\mathrm{CoS}^p_{\mathcal{W}_0 B}((\mathbf{L}\mathcal{W}_1\Omega^1_{B/R})_{[-k]}),\mathcal{W}_0 B),\delta+[\varpi,-])_{[-k]}$$

which is seldom acyclic. This is similar to phenomena arising in [Pri1, Pri2, Pri3], where the only obstruction to quantisation is first-order, and can be eliminated by restricting to involutive quantisations.

If ϖ is non-degenerate in the sense that $\mathbf{L}\Omega^1_{\mathcal{W}_0 B/R}$ is perfect over $\mathcal{W}_B$ and that the Lie bracket ϖ induces a quasi-isomorphism $(\mathbf{L}\Omega^1_{\mathcal{W}_0 B/R})_{[1-k]} \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{W}_0 B}(\mathbf{L}\mathcal{W}_1\Omega^1_{B/R}, \mathcal{W}_0 B)$, then as in [Pri4, Definition 1.16 and Lemma 1.17], there are $\mathbb{G}_m$-equivariant CDGA quasi-isomorphisms

$$\bigoplus_{p\geqslant 0}(\mathbf{L}\Omega^p_{\mathcal{W}_0 B/R})_{[p]} \to \bigoplus_{p\geqslant 0}\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{W}_0 B}(\mathbf{L}\mathrm{CoS}^p_{\mathcal{W}_0 B}((\mathbf{L}\mathcal{W}_1\Omega^1_{B/R})_{[-k]}), \mathcal{W}_0 B)$$

given on generators by the identity on B and $df \mapsto [\varpi, f]$. Moreover, this map sends the de Rham differential d to the differential $[\varpi, -]$, so $\mathcal{W}_{-1}\mathbf{R}\mathcal{D}er_{P_k^{ac},R}(B)$ is then quasi-isomorphic to the truncated de Rham complex $(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0 B/R})^{[k]} := (\mathrm{Tot}^{\Pi}\mathbf{L}\Omega^{\geqslant 1}_{\mathcal{W}_0 B/R})^{[k]}$. In particular, equivalence classes of deformations of non-involutive deformations are parametrised by $\mathrm{H}^{k+1}(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0 B/R})$ under these conditions.

Remark 1.19. Following Remark 1.13, if we take an involutively filtered dg operad $(\mathcal{P}, W)$ with an involutive equivalence $\mathrm{gr}^W\mathcal{P} \simeq P_k^{ac}$, then the conditions of Theorem 1.17 also ensure that the space of almost commutative involutive derived $\mathcal{P}$-algebras (B', W) with $\mathrm{gr}^W B' \simeq B$ is contractible. In particular, when $k = 1$ we can take $\mathcal{P}$ to be the Beilinson–Drinfeld operad, given by the PBW filtration on the associative operad, to see that there is an essentially unique filtered associative dg algebra (B', W) equipped with a filtered involution $(B')^{\mathrm{opp}} \cong B'$ and an equivalence $\mathrm{gr}^W B' \simeq B$. When B is an algebra of polyvectors, B' will thus be given by the ring of differential operators $\mathscr{D}(\omega^{\frac{1}{2}})$ on a square root $\omega^{\frac{1}{2}}$ of the dualising bundle whenever this exists, and gives a ring of twisted differential operators generalising $\mathscr{D}(\omega^{\frac{1}{2}})$ even when the dualising complex is not a line bundle, or has no square root.

As in Remark 1.18, we could also consider the space of almost commutative derived $\mathcal{P}$-algebras (B', W) with $\mathrm{gr}^W B' \simeq B$, and find this is governed by the abelian DGLA $(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0 B/R})^{[k]}$ when ϖ is non-degenerate, so equivalence classes of non-involutive deformations are again a torsor for $\mathrm{H}^{k+1}(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0 B/R})$. The same reasoning applies for non-involutively filtered dg operads $(\mathcal{P}, W)$ with an involutive equivalence $\mathrm{gr}^W\mathcal{P} \simeq P_k^{ac}$, except that there is then a potentially non-zero obstruction class in $\mathrm{H}^{k+2}(\mathrm{Tot}^{\Pi}\mathbf{L}F^1\Omega^\bullet_{\mathcal{W}_0 B/R})$ to such algebras existing.

2. Quantisations on derived stacks

2.1. **Hochschild complexes.** We now recall some constructions used in [Pri3]. We say that a complete filtered DGAA (A, F) is almost commutative if $\mathrm{gr}_F A$ is a CDGA. The following are abbreviated versions of [Pri3, Definitions 1.7 and 1.15]

Definition 2.1. We write B for the bar construction from possibly non-unital DGAAs over R to ind-conilpotent differential graded associative coalgebras (DGACs) over R. Explicitly, this is given by taking the tensor coalgebra

$$\mathrm{B}A := T(A_{[-1]}) = \bigoplus_{i\geqslant 0}(A_{[-1]})^{\otimes_R i},$$

with chain differential given on cogenerators $A_{[-1]}$ by combining the chain differential and multiplication on A. Write B_+A for the subcomplex $T_+(A_{[-1]}) = \bigoplus_{i>0} A_{[-1]}^{\otimes_R i}$.

Let Ω_+ be the left adjoint to B_+.

Definition 2.2. For an almost commutative DGAA (A,F) over R and a filtered (A,F)-bimodule (M,F) in chain complexes for which the left and right $\mathrm{gr}^F A$-module structures on $\mathrm{gr}^F M$ agree, we define the filtered chain complex

$$\mathrm{CC}_{R,BD_1}(A,M)$$

to be the completion of the cohomological Hochschild complex $\mathrm{CC}_R(A,M)$ (rewritten as a chain complex) with respect to the filtration γF defined as follows. We may identify $\mathrm{CC}_R(A,M)$ with the subcomplex of

$$\underline{\mathrm{Hom}}_R(\mathrm{B}A, \mathrm{B}(A \oplus M_{[1]}))$$

consisting of coderivations extending the zero coderivation on $\mathrm{B}A$. The hypotheses on M ensure that $A \oplus M$ is almost commutative (regarding M as a square-zero ideal), so we have filtrations βF on $\mathrm{B}A$ and $\mathrm{B}(A \oplus M_{[1]})$. We then define $(\gamma F)_i$ to consist of coderivations sending $(\beta F)_j \mathrm{B}A$ to $(\beta F)_{i+j-1}\mathrm{B}(A \oplus M)$. We also define the subcomplex $\mathrm{CC}_{R,BD_1,+}(A,M)$ to be the kernel of $\mathrm{CC}_{R,BD_1}(A,M) \to M$, or equivalently $\underline{\mathrm{Hom}}_R(\mathrm{B}_+A, M)^{\#}$. We write $\mathrm{CC}_{R,BD_1}(A) := \mathrm{CC}_{R,BD_1}(A,A)$.

The following are taken from [Pri3, §1.2.1]

Definition 2.3. Given a brace algebra B, define the opposite brace algebra B^{opp} to have the same elements as B, but multiplication $b^{\mathrm{opp}} \smile c^{\mathrm{opp}} := (-1)^{\deg b \deg c}(c \smile b)^{\mathrm{opp}}$ and brace operations given by the multiplication $(\mathrm{B}B^{\mathrm{opp}}) \otimes (\mathrm{B}B^{\mathrm{opp}}) \to \mathrm{B}B^{\mathrm{opp}}$ induced by the isomorphism $(\mathrm{B}B^{\mathrm{opp}}) \cong (\mathrm{B}B)^{\mathrm{opp}}$. Explicitly,

$$\{b^{\mathrm{opp}}\}\{c_1^{\mathrm{opp}}, \ldots, c_m^{\mathrm{opp}}\} := \pm\{b\}\{c_m, \ldots, c_1\}^{\mathrm{opp}},$$

where $\pm = (-1)^{m(m+1)/2 + (\deg f - m)(\sum_i \deg c_i - m) + \sum_{i<j} \deg c_i \deg c_j}$.

Observe that when a filtered brace algebra B is almost commutative, then so is B^{opp}.

Definition 2.4. Define a filtration γ on the brace operad Br of [Vor] by putting the cup product in γ_0 and the braces $\{-\}\{-, \ldots, -\}_r$ in γ_{-r}.

Thus a (brace, γ)-algebra (A,F) in filtered complexes is a brace algebra for which the cup product respects the filtration, and the r-braces send F_i to F_{i-r}. We refer to (brace, γ)-algebras as almost commutative brace algebras.

We define an almost commutative involutive brace algebra to be an almost commutative brace algebra (A,F) equipped with an involution $(A,F) \cong (A^{\mathrm{opp}}, F)$ of brace algebras which acts on $\mathrm{gr}_i^F A$ as multiplication by $(-1)^i$.

Lemma 2.5. *Given almost commutative DGAAs A, D over R, following* [Bra, §2.1] *there is an involution*

$$-i\colon\ \mathrm{CC}_{R,BD_1}(A,D)^{\mathrm{opp}} \to \mathrm{CC}_{R,BD_1}(A^{\mathrm{opp}}, D^{\mathrm{opp}})$$

of DGAAs given by

$$i(f)(a_1, \ldots, a_m) = -(-1)^{\sum_{i<j} \deg a_i \deg a_j}(-1)^{m(m+1)/2} f(a_m^{\mathrm{opp}}, \ldots, a_1^{\mathrm{opp}})^{\mathrm{opp}}.$$

When $A = D$, the involution $-i$ makes $(\mathrm{CC}_{R,BD_1}(A), \gamma F)$ into an almost commutative involutive brace algebra.

2.2. **Involutions from the Grothendieck–Teichmüller group.** The good truncation filtration τ on the brace operad is contained in the filtration γ on Br from Definition 2.4, so the quasi-isomorphism in in [Vor] between the brace operad Br and the $\mathbb{Q}$-linear operad $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ of chains on the topological operad E_2 induces filtered quasi-isomorphisms

$$(\mathrm{C}_\bullet(E_2, \mathbb{Q}), \tau) \to (\mathrm{Br}, \tau) \to (\mathrm{Br}, \gamma).$$

As observed in [Pri2, Remark 2.21], the involution of the brace operad in Definition 2.3 corresponds under this quasi-isomorphism to an involution of the E_2 operad, which takes an embedding $[1, k] \times I^2 \to I^2$ of k little squares in a big square, and reverses the order of the labels $[1, k]$ with appropriate signs.

As summarised in the preamble to the theorem in [Pet], the space of homotopy automorphisms of the coloured operad given by rationalisation of E_2 is homotopy equivalent the Grothendieck–Teichmüller group $\mathrm{GT}(\mathbb{Q})$. Thus our involution comes from an element $t \in \mathrm{GT}(\mathbb{Q})$ which lies over $-1 \in \mathbb{G}_m(\mathbb{Q})$; in other words, t is a (-1)-Drinfeld associator.

Definition 2.6. Denote the pro-unipotent radical of the pro-algebraic group GT by GT^1. Write $\mathrm{Levi}_{\mathrm{GT}}$ for the space of Levi decompositions $\mathrm{GT} \cong \mathbb{G}_m \ltimes \mathrm{GT}^1$, or equivalently of sections of the natural map $\mathrm{GT} \to \mathbb{G}_m$. We then define $\mathrm{Levi}^t_{\mathrm{GT}}$ to be the space of sections w of $\mathrm{GT} \to \mathbb{G}_m$ satisfying $w(-1) = t$.

Taking base change to arbitrary commutative $\mathbb{Q}$-algebras A gives sets $\mathrm{Levi}_{\mathrm{GT}}(A), \mathrm{Levi}^t_{\mathrm{GT}}(A)$ of decompositions over A.

Lemma 2.7. *The functor $\mathrm{Levi}^t_{\mathrm{GT}}$ is an affine scheme over $\mathbb{Q}$ equipped with the structure of a trivial torsor for the subgroup scheme $(\mathrm{GT}^1)^t \subset \mathrm{GT}^1$ given by the centraliser of t.*

Proof. We expand the argument from [Pri2, Remark 2.21]. By the general theory [HM] of pro-algebraic groups in characteristic 0, the set $\mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ is non-empty, and for all commutative $\mathbb{Q}$-algebras A, the group $\mathrm{GT}^1(A)$ acts transitively on $\mathrm{Levi}_{\mathrm{GT}}(A)$ via the adjoint action. Because the graded quotients of the lower central series of GT^1 have non-zero weight for the adjoint $\mathbb{G}_m$-action, the centralisers of this action are trivial and $\mathrm{Levi}_{\mathrm{GT}}(A)$ is a torsor for $\mathrm{GT}^1(A)$.

Now, choose any Levi decomposition $w_0 \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ and let $w_0(-1) = tu$ for $u \in \mathrm{GT}^1(\mathbb{Q})$. Since t and $w_0(-1)$ are both of order 2, we have $u = \mathrm{ad}_t(u^{-1})$. Writing $u = \exp(v)$ and $u^{\frac{1}{2}} := \exp(\frac{1}{2}v)$, we have $u^{\frac{1}{2}} = \mathrm{ad}_t(u^{-\frac{1}{2}})$, giving $w := \mathrm{ad}_{u^{-\frac{1}{2}}} \circ w_0 \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$. Thus $\mathrm{Levi}^t_{\mathrm{GT}}$ is non-empty, so $\mathrm{Levi}^t_{\mathrm{GT}} \subset \mathrm{Levi}_{\mathrm{GT}}$ is a torsor for the subgroup $(\mathrm{GT}^1)^t \subset \mathrm{GT}^1$ fixing t under the adjoint action. □

As explained succinctly in [Pet], formality of the operad $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ is a consequence of the observation that the Grothendieck–Teichmüller group is a pro-unipotent extension of $\mathbb{G}_m$. Since GT acts on the operad $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ of chains, any Levi decomposition $w\colon \mathbb{G}_m \to \mathrm{GT}$ gives a weight decomposition (i.e. a $\mathbb{G}_m$-action) of $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ which splits the good truncation filtration τ, so gives an equivalence between $\mathrm{C}_\bullet(E_2, \mathbb{Q})$ and P_2 respecting the natural map from the Lie operad. Since this equivalence necessarily preserves the good truncation filtrations τ, it also gives an equivalence θ_w between $(P_2, \tau) = (\mathrm{H}_*(E_2, \mathbb{Q}), \tau)$ and $(\mathrm{C}_\bullet(E_2, \mathbb{Q}), \tau) \simeq (\mathrm{Br}, \gamma)$.

Note that a Levi decomposition $w \in \mathrm{Levi}_{\mathrm{GT}}(k)$ is determined by the λ-associator $w(\lambda)$ for any $\lambda \in k^\times$ not a root of unity. This associator commutes with t whenever

$w \in \mathrm{Levi}^t_{\mathrm{GT}}(k)$, since $w\mathbb{G}_m$ is its own centraliser, and $w(-1)$ is the only element of order 2 it contains.

Definition 2.8. Given a Levi decomposition $w \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$, we denote by p_w the ∞-functor from almost commutative brace algebras to almost commutative P_2-algebras coming from the equivalence $\theta_w \colon (P_2, \tau) \simeq (\mathrm{Br}, \gamma)$ induced by w. This preserves the underlying filtered L_∞-algebras up to equivalence.

The pro-unipotent radical $\mathrm{GT}^1 \subset \mathrm{GT}$ acts trivially on homology $\mathrm{H}_*(E_2, \mathbb{Q}) \cong P_2$, inducing a $\mathbb{G}_m$-action on P_2. Since the $\mathbb{G}_m$-action on $\mathrm{H}_1(E_2(2), \mathbb{Q})$ has weight 1, this gives a $\mathbb{G}_m$-equivariant isomorphism

$$\mathrm{H}_*(E_2, \mathbb{Q}) \cong P_2^{ac}$$

for the $\mathbb{G}_m$-equivariant operad $P_2^{ac} = \mathrm{Com} \circ s^{-1}\hbar\mathrm{Lie}$ of Definition 1.6.

The element $t \in \mathrm{GT}(\mathbb{Q})$ lies over $-1 \in \mathbb{G}_m(\mathbb{Q})$, so the involution t on E_2 induces the action of $-1 \in \mathbb{G}_m$ on P_2^{ac} under the isomorphism above.

Definition 2.9. When $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$, the equivalence $\theta_w \colon (P_2, \tau) \simeq (\mathrm{Br}, \gamma)$ commutes with the involution t, so it induces an equivalence between almost commutative involutive brace algebras and almost commutative involutive P_2-algebras, which we also denote by p_w.

2.3. **Quantisations on derived Deligne–Mumford stacks.**

Theorem 2.10. *Given a morphism $R \to A$ of CDGAs with perfect cotangent complex $\mathbf{L}\Omega^1_{A/R}$ and $A_\#$ flat over $R_\#$, the filtered DGLA underlying the Hochschild complex*

$$\mathrm{CC}_{R,BD_1}(A)_{[-1]}$$

is quasi-isomorphic to the graded DGLA

$$\mathrm{Pol}(A/R, 0)_{[-1]} := \bigoplus_{p \geqslant 0} \mathbf{L}\Lambda^p_A(T_{A/R})_{[p-1]}$$

of derived polyvectors on A, where $T_{A/R} := \mathbf{R}\underline{\mathrm{Hom}}_A(\mathbf{L}\Omega^1_{A/R}, A)$ is the derived tangent space, and the Lie algebra structure is given by the Schouten–Nijenhuis bracket.

This quasi-isomorphism depends only on a choice of Levi decomposition $w \in \mathrm{Levi}^t_{\mathrm{GT}}$, and is functorial with respect to formally étale morphisms.

Proof. Lemma 2.5 shows that $\mathrm{CC}_{R,BD_1}(A)$ is an almost commutative involutive brace algebra, and the Poincaré–Birkhoff—Witt isomorphism gives $\mathrm{gr}^\gamma\mathrm{CC}_{R,BD_1}(A) \simeq \mathrm{Pol}(A/R, 0) =: P$. Applying the ∞-functor p_w of Definition 2.9 for some $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$ (or even a point in the space $\mathrm{Levi}^t_{\mathrm{GT}}(R)$) gives an almost commutative involutive P_2-algebra $p_w\mathrm{CC}_{R,BD_1}(A)$ with associated graded P, as a $\mathbb{G}_m$-equivariant P_2^{ac}-algebra.

The $\mathbb{G}_m$-equivariant P_2^{ac}-algebra P over R is non-negatively weighted, and $\mathbf{L}\Omega^1_{P/A}$ is freely generated by the module $(T_{A/R})_{[1]}$, which has weight 1. It thus satisfies the conditions of Theorem 1.17, giving an essentially unique equivalence

$$\alpha_{w,A} \colon p_w\mathrm{CC}_{R,BD_1}(A) \simeq \mathrm{Pol}(A/R, 0)$$

of almost commutative involutive P_2-algebras. In particular, there exists a zigzag of filtered quasi-isomorphisms between the underlying DGLAs.

Finally, if D denotes the [1]-diagram $(A \xrightarrow{f} B)$ with f formally étale, then as in [Pri2, §3.1], we have an almost commutative involutive brace algebra $\mathrm{CC}_{R,BD_1}(D)$ with restriction maps

$$\mathrm{CC}_{R,BD_1}(A) \xleftarrow{\sim} \mathrm{CC}_{R,BD_1}(D) \to \mathrm{CC}_{R,BD_1}(B),$$

the left-hand map being a filtered quasi-isomorphism, and the associated graded of the right-hand map being formally étale. In the ∞-category of almost commutative brace algebras, we thus have a map $\phi_f \colon \mathrm{CC}_{R,BD_1}(A) \to \mathrm{CC}_{R,BD_1}(B)$ inducing a formally étale map $\mathrm{gr}^\gamma\phi_f \colon \mathrm{Pol}(A/R,0) \to \mathrm{Pol}(B/R,0)$ of $\mathbb{G}_m$-equivariant P_2^{ac}-algebras on the associated gradeds. Theorem 1.17 thus provides an essentially unique homotopy $\alpha_{w,B} \circ \phi_f \circ \alpha_{w,A}^{-1} \simeq \mathrm{gr}_\gamma\phi_f$, giving functoriality. □

Remark 2.11. When applied to polynomial rings, the statement of Theorem 1.17 recovers [Kon1, Theorem 4], and for more general smooth algebraic varieties it recovers [VdB, Theorem 1.1]. The preliminary steps are the same, but the arguments for eliminating the potential first-order deformation are very different, as we consider involutive deformations where Kontsevich looks at invariance under affine transformations.

The hypotheses of the following corollary are satisfied by any derived Deligne–Mumford stack locally of finite presentation over the CDGA R. When $R = \mathrm{H}_0R$, this includes underived schemes which are local complete intersections over R, in which case the cotangent complex $\mathbf{L}\Omega^1_{X/R}$ is concentrated in homological degrees $[0,1]$.

Corollary 2.12. *Given a derived DM N-stack $\mathfrak{X}$ over R with perfect cotangent complex $\mathbf{L}\Omega^1_{\mathfrak{X}/R}$, the space $Q\mathcal{P}(\mathfrak{X},0)$ of E_1 quantisations of $\mathfrak{X}$ from* [Pri2, Definitions 1.23, 3.9] *is equivalent to the Maurer–Cartan space*

$$\underline{\mathrm{MC}}(\mathbf{R}\Gamma(\mathfrak{X}, (\hbar(\mathscr{O}_{\mathfrak{X}})_{[-1]} \times \hbar T_{\mathfrak{X}/R} \times \prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1})[\![\hbar]\!])).$$

In particular, there exists a quantisation for every Poisson structure

$$\pi \in \mathcal{P}(\mathfrak{X},0) = \underline{\mathrm{MC}}(\mathbf{R}\Gamma(\mathfrak{X}, \prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1})),$$

in the form of a curved A_∞-deformation of $\mathscr{O}_{\mathfrak{X}}$.

Proof. The space $Q\mathcal{P}(\mathfrak{X},0)$ of quantisations is defined to be $\underline{\mathrm{MC}}(\prod_{j\geqslant 2}\mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \gamma_j\mathrm{CC}_R(\mathscr{O}_{\mathfrak{X}})_{[-1]}\hbar^{j-1})$, and Theorem 2.10 allows us to substitute for γ_j to give an equivalence between this and the space above for each $w \in \mathrm{Levi}^t_{\mathrm{GT}}(\mathbb{Q})$. Existence of quantisations for a Poisson structure π then follows by observing that extension by zero gives a DGLA morphism

$$\prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1} \to (\hbar(\mathscr{O}_{\mathfrak{X}})_{[-1]} \times \hbar T_{\mathfrak{X}/R} \times \prod_{p\geqslant 2} \mathbf{L}\Lambda^p_{\mathscr{O}_{\mathfrak{X}}}(T_{\mathfrak{X}/R})_{[p-1]}\hbar^{p-1})[\![\hbar]\!].$$

□

Remark 2.13. Replacing Hochschild complexes with complexes of polydifferential operators should allow these results to generalise to derived differential geometry. Complexes of smooth polyvectors will be freely generated as a CDGA over the ring of smooth functions by the shifted smooth tangent complex. This will satisfy the conditions of Theorem 2.10 even though the smooth tangent complex is much smaller than the tangent complex of the ring of derived $\mathcal{C}^\infty$ functions as an abstract CDGA.

On the other hand, for derived Artin stacks, quantisations and Poisson structures are defined in [Pri2, Pri4] in terms of semi-infinite total complexes arising from doubly graded differential algebras. Since passing to total complexes does not preserve derived symmetric powers, having a perfect cotangent complex does not seem to guarantee that the conditions of Theorem 2.10 are then satisfied by complexes of multiderivations in this setting. We would instead need to retain some of the stacky structure, constraining $\eth$ so that it acts as an element of the semi-infinite total Hom complex from [Pri2]. In order to apply formality theorems, this means we would probably have to realise the Hochschild complex as an E_2-algebra in some category of BV-algebras as suggested in [Pri3, Remarks 5.4], with the BV operator Δ constrained to be a stacky differential operator.

References

[BM] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003. arXiv:math/0206094 [math.AT].

[Bra] Christopher Braun. Involutive A_∞-algebras and dihedral cohomology. *J. Homotopy Relat. Struct.*, 9(2):317–337, 2014. arXiv:1209.1261v2 [math.QA].

[Cav] G. Caviglia. A Model Structure for Enriched Coloured Operads. arXiv:1401.6983 [math.AT], 2014.

[CPT$^+$] D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. Shifted Poisson structures and deformation quantization. *J. Topol.*, 10(2):483–584, 2017. arXiv:1506.03699v4 [math.AG].

[Fre] Benoit Fresse. *Homotopy of operads and Grothendieck-Teichmüller groups. Part 2*, volume 217 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. The applications of (rational) homotopy theory methods.

[Hin1] Vladimir Hinich. Homological algebra of homotopy algebras. *Comm. Algebra*, 25(10):3291–3323, 1997. arXiv:q-alg/9702015v1, erratum arXiv:math/0309453v3 [math.QA].

[Hin2] Vladimir Hinich. DG coalgebras as formal stacks. *J. Pure Appl. Algebra*, 162(2-3):209–250, 2001.

[HM] G. Hochschild and G. D. Mostow. Pro-affine algebraic groups. *Amer. J. Math.*, 91:1127–1140, 1969.

[Kon1] Maxim Kontsevich. Operads and motives in deformation quantization. *Lett. Math. Phys.*, 48(1):35–72, 1999. Moshé Flato (1937–1998).

[Kon2] Maxim Kontsevich. Deformation quantization of algebraic varieties. *Lett. Math. Phys.*, 56(3):271–294, 2001. EuroConférence Moshé Flato 2000, Part III (Dijon).

[LV] Jean-Louis Loday and Bruno Vallette. *Algebraic operads.* Berlin: Springer, 2012.

[Pet] Dan Petersen. Minimal models, GT-action and formality of the little disk operad. *Selecta Math. (N.S.)*, 20(3):817–822, 2014. arXiv:1303.1448v1 [math.AT].

[Pri1] J. P. Pridham. Deformation quantisation for (-1)-shifted symplectic structures and vanishing cycles. arXiv:1508.07936v2 [math.AG], 2015.

[Pri2] J. P. Pridham. Deformation quantisation for unshifted symplectic structures on derived Artin stacks. arXiv: 1604.04458v2 [math.AG], 2016.

[Pri3] J. P. Pridham. Quantisation of derived Lagrangians. arXiv: 1607.01000v1 [math.AG], 2016.

[Pri4] J. P. Pridham. Shifted Poisson and symplectic structures on derived N-stacks. *J. Topol.*, 10(1):178–210, 2017. arXiv:1504.01940v5 [math.AG].

[Tam] Dmitry E. Tamarkin. *Operadic proof of M. Kontsevich's formality theorem.* ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–The Pennsylvania State University.

[Toë] Bertrand Toën. Derived algebraic geometry and deformation quantization. In *Proceedings of the International Congress of Mathematicians (Seoul 2014), Vol. II*, pages 769–752, 2014. arXiv:1403.6995v4 [math.AG].

[VdB] Michel Van den Bergh. On global deformation quantization in the algebraic case. *J. Algebra*, 315(1):326–395, 2007.

[Vor] Alexander A. Voronov. Homotopy Gerstenhaber algebras. In *Conférence Moshé Flato 1999, Vol. II (Dijon)*, volume 22 of *Math. Phys. Stud.*, pages 307–331. Kluwer Acad. Publ., Dordrecht, 2000. arXiv:9908040 [math.QA].